Complex Analysis qualification exam: May 2020

Note: all statements require proofs. You can make references to standard theorems from the course; however, you need state the relevant part of the theorem in your own words, unless it's a well known named theorem. For example, "we had a theorem in the class that said that any continuous function on a compact subset of $\mathbb{R}^n$ is uniformly continuous" is a good reference, and "by the uniqueness theorem from the class, f is unique" is not a good reference.

(1) Let $u(z) = u(x + iy) = \log(x^2 + y^2)$ (here $x, y \in \mathbb{R}$ and the usual real logarithm is used). Find a holomorphic function $f\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ such that $u = \operatorname{Re} f$, or explain why it does not exist.

(2) Find a conformal isomorphism between the set $\{|z| < 1, |z - i| < 1\}$ and the upper half-plane.

(3) Does the function $f(z) = \sqrt{z(z-1)}$ have a holomorphic branch in a neighborhood of $z = \infty$ (that is, in some set $\{R < |z| < +\infty\}$). If yes, find its principal part around $z = \infty$. In other words, find a polynomial $p(z)$ such that $f(z) - p(z)$ has a removable singularity at $z = \infty$.

(4) Find $\oint_C \frac{dz}{\sqrt{z^2+z+1}}$, where C is a circle $|z| = r$ with $r \neq 1$, oriented counter-clockwise. Use any branch of $\sqrt{\cdot}$, but specify which one you are using.

(5) Find $\int_0^\pi \tan(x + ia)\, dx$ for $a \in \mathbb{R}$. If the integral does not converge absolutely, find its principal value.